COMPUTING GENERALIZED FROBENIUS POWERS OF MONOMIAL IDEALS

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Abstract. Generalized Frobenius powers of an ideal were introduced in [HTW18] as characteristic-dependent analogs of test ideals. However, little is known about the Frobenius powers and critical exponents of specific ideals, even in the monomial case. We describe an algorithm to compute the critical exponents of monomial ideals, and use this algorithm to prove some results about their Frobenius powers and critical exponents. Rather than using test ideals, our algorithm uses techniques from linear optimization.

1. Introduction

Frobenius powers of an ideal with non-negative real-valued exponents were introduced in [HTW18] as a characteristic-dependent analog of test ideals in F-finite regular domains of prime characteristic. The motivation for defining Frobenius powers was to find a prime characteristic invariant sensitive enough to mimic a property of multiplier ideals in characteristic zero, namely that the multiplier ideal $\mathcal{J}(I^\lambda)$ agrees with the multiplier ideal $\mathcal{J}(f^\lambda)$ for a general $f \in I$. In particular, it is known that $\mathcal{J}(f^\lambda) = \mathcal{J}(I^\lambda)$ when I is the monomial ideal generated by the terms of f [How03]. Thus computations for arbitrary ideals can be reduced to the monomial case.

Let $S = K[x_1, \dots, x_m]$ be a standard graded polynomial ring over a field K of characteristic $p > 0$. We let $F : S \to S$ denote the Frobenius homomorphism of S. Recall that a ring S of characteristic p is called *F-finite* if S is a finitely generated as a module over the subring $F(S) = S^p = \{f^p : f \in S\}$. We will assume throughout that K is an F-finite field so that the polynomial ring S is also F-finite. However, we will quickly see there is no loss of generality in assuming $K = \mathbb{F}_p$.

The main result of our paper is Algorithm 3.7/Theorem 3.12, a deterministic algorithm which computes all the critical exponents, and hence all the fractional Frobenius powers, of an arbitrary monomial ideal over K. This algorithm does not involve test elements; instead, it uses analytic geometry, base p arithmetic, and a generalization to vectors of the long division algorithm for computing the decimal expansion of a fraction. The algorithm appears to be very efficient in small characteristic. As an immediate corollary to the algorithm, we recover by methods that avoid any reference to test ideals and F-thresholds that the critical exponents of monomial ideals are rational [HTW18, 5.8].

Section 2 recalls necessary background on Frobenius powers and arithmetic in base p. It also introduces notation that will be used throughout the paper, including the (p, n)-Sierpinski simplex, a fractal that we use to describe the Frobenius powers. Section 3 describes the algorithm for computing the critical exponents of a monomial ideal and gives the proof of its validity. The algorithm is worked out in some detail for the ideal (x^2y^2, x^3z^3) over $\mathbb{F}_3$ and $\mathbb{F}_5$ in Examples 3.9 and 3.10.

In Section 4, we use our techniques to compute some Frobenius powers and critical exponents in more generality. Example 4.2 does this computation for (x^2y^2, x^3z^3) in all characteristics simultaneously, demonstrating as a proof-of-concept that this is possible. Proposition 4.1 computes the least critical exponent for any height one monomial ideal containing a pure power. We close with some questions for further research.

2. Background on Frobenius Powers

2.1. **Frobenius Powers.** Let S be an F-finite standard graded polynomial ring as above. In [HTW18], Hernández-Teixeira-Witt define the *Frobenius powers* of an ideal $I \subseteq S$ as a family of ideals $I^{[\lambda]}$ parametrized by a non-negative real number λ, which agree with the usual Frobenius powers $I^{[p^e]} = (f^{p^e} \mid f \in I)$ when $\lambda = p^e$. As one might hope, this family of ideals has good containment properties:

Proposition 2.1 ([HTW18, 3.16])**.** *Let $I, J \subseteq S$ be ideals, and let $\lambda, \mu \in \mathbb{R}_{\geq 0}$. Then:*

(a) (Monotonicity) *If $\lambda < \mu$, then $I^{[\lambda]} \supseteq I^{[\mu]}$.*
(b) (Right Constancy) *For every λ, there exists an $\varepsilon > 0$ such that $I^{[\mu]} = I^{[\lambda]}$ whenever $\lambda \leq \mu < \lambda + \varepsilon$.*
(c) $I^{[\lambda]}I^{[\mu]} \supseteq I^{[\lambda+\mu]}$
(d) *For any ideal $J \subseteq S$, we have $I^{[\lambda]}J^{[\lambda]} \supseteq (IJ)^{[\lambda]}$.*

Similar to jumping numbers and F-jumping numbers of multiplier ideals and test ideals, we are interested in determining the Frobenius powers of various monomial ideals I and the real numbers $\lambda > 0$ such that $I^{[\mu]} \neq I^{[\lambda]}$ for all $\mu < \lambda$, which are called the *critical exponents* of I. In particular, there is a smallest positive critical exponent by right constancy; it is called the *least critical exponent* of I and is denoted by $\operatorname{lce}(I)$.

In the remainder of this subsection, we summarize the stages by which generalized Frobenius powers are constructed, and we make some simple observations that simplify the case of monomial ideals to working over $\mathbb{F}_p$.

Given an ideal $I \subseteq S$ and $\lambda \in \mathbb{R}_{\geq 0}$, the Frobenius power $I^{[\lambda]}$ is constructed as follows:

- If $\lambda = k$ is an integer with base p expansion $k = k_0 + k_1 p + \cdots k_r p^r$, then
$$I^{[k]} = I^{k_0}(I^{k_1})^{[p]} \cdots (I^{k_r})^{[p^r]}$$
- If $\lambda = \frac{k}{q} \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}$ is a non-negative p-adic rational, we define $I^{[\frac{k}{q}]} = (I^{[k]})^{[\frac{1}{q}]}$, where for any ideal J, the ideal $J^{[1/q]}$ is the smallest ideal L such that $L^{[q]} \supseteq J$ as originally defined in [BMS08]. In practice, because we are ultimately interested in ideals J in a polynomial ring over $\mathbb{F}_p$, the ideal $J^{[\frac{1}{q}]}$ is always easily computable by [BMS08, 2.5].
- For any real number $\lambda \geq 0$, the Frobenius power $I^{[\lambda]}$ is then defined by taking any monotone decreasing sequence (λ_j) of p-adic rationals converging to λ from above. The monotonicity of Frobenius powers then yields an ascending chain of ideals $I^{[\lambda_1]} \subseteq I^{[\lambda_2]} \subseteq \cdots$, and $I^{[\lambda]}$ is defined to be the stable value of this chain, which exists since S is Noetherian.

In particular, every real Frobenius power is the Frobenius power of some p-adic rational.

Proposition 2.2. *Let $\varphi : S \to T$ be a ring homomorphism between F-finite regular domains, $I \subseteq S$ be an ideal, and $\lambda \in \mathbb{R}_{\geq 0}$. Then:*

(a) *$(IT)^{[\lambda]} \subseteq I^{[\lambda]}T$, with equality if λ is an integer.*
(b) *If in addition S is free over S^q with basis $e_1, \ldots, e_n$, T is free over T^q, and $\varphi(e_1), \ldots, \varphi(e_n)$ are part of a basis for T over T^q, then $(IT)^{[\frac{k}{q}]} = I^{[\frac{k}{q}]}T$.*

Proof. (a) If $\lambda = k$ is an integer, it is clear that $(IT)^{[k]} = I^{[k]}T$ since homomorphisms preserve both ordinary powers and p-th power Frobenius powers. As $(I^{[\frac{k}{q}]})^{[q]} \supseteq I^{[k]}$, we have $(I^{[\frac{k}{q}]}T)^{[q]} = (I^{[\frac{k}{q}]})^{[q]}T \supseteq I^{[k]}T = (IT)^{[k]}$ so that $I^{[\frac{k}{q}]}T \supseteq (IT)^{[\frac{k}{q}]}$. The claim then follows for arbitrary λ by applying the preceding inclusions to a monotone decreasing sequence of p-adic rationals converging to λ.

(b) By the previous part, it is enough to show that $I^{[\frac{1}{q}]}T = (IT)^{[\frac{1}{q}]}$ for any ideal $I \subseteq S$. For $f \in I$, write $f = \sum_i f_i^q e_i$, so $\varphi(f) = \sum_i \varphi(f_i)^q \varphi(e_i)$. By [BMS08, 2.5], both the ideals $(IT)^{[\frac{1}{q}]}$ and $I^{[\frac{1}{q}]}T$ are generated by all elements of the form $\varphi(f_i)$ for some $f \in I$. $\square$

Remark 2.3. It is worth noting two important cases in which one can apply the second part of Proposition 2.2 are when:

- T is obtained from S by extension of the ground field. Thus, there is no loss of generality by working over $\mathbb{F}_p$ when computing the Frobenius powers of monomial ideals.
- $S = \mathbb{F}_p[x_1, \ldots, x_m]$, $T = \mathbb{F}_p[y_1, \ldots, y_s]$, and $\varphi(x_1), \ldots, \varphi(x_m)$ are square-free monomials with disjoint supports. In this case, S has a well-known basis over S^q consisting of all monomials $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_m^{a_m}$ with $a_i < q$ for

all i, and $\varphi(\mathbf{x}^{\mathbf{a}}) = \mathbf{y}^{\mathbf{b}}$ for some $\mathbf{b} \in \mathbb{N}^m$ with $b_j < q$ for all j by assumption so that $\varphi(\mathbf{x}^{\mathbf{a}})$ is part of the monomial basis for T over T^q.

2.2. **Frobenius Powers of Monomial Ideals.** In this subsection, we fix the notation used throughout the rest of the paper and make some simple observations about the Frobenius powers of monomial ideals.

Notation 2.4. If $\mathbf{x}^{\mathbf{b}} = x_1^{b_1} \dots x_m^{b_m}$ is a monomial of S, we say that $\mathbf{b} = (b_1, \dots, b_m) \in \mathbb{N}^m$ is the *exponent vector* of $\mathbf{x}^{\mathbf{b}}$. Henceforth, we assume that $I = (\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_n})$ is a nonzero, proper monomial ideal in S, and we let $A = (\mathbf{a}_1 | \cdots | \mathbf{a}_n)$ be the $m \times n$ matrix whose columns are the exponent vectors of the generating monomials of I. We set $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^n$. All norms of vectors $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ refer to the 1-norm $\|\mathbf{v}\| = \sum_i |v_i|$. In particular, if $\mathbf{u} \in \mathbb{N}^n$ and $k \in \mathbb{N}$, we recall that the *multinomial coefficient* $\binom{k}{\mathbf{u}}$ is equal to $\frac{k!}{u_1! u_2! \cdots u_n!}$ if $\|\mathbf{u}\| = \sum_i u_i = k$ and is equal to zero otherwise.

Convention 2.5. We adapt operations/relations on numbers to vectors $\mathbf{u} \in \mathbb{R}^n$ coordinatewise. For example, $\lfloor \mathbf{u} \rfloor = (\lfloor u_1 \rfloor, \dots, \lfloor u_n \rfloor)$ is the vector obtained by taking the floor of each component; we write $\mathbf{u} \le \mathbf{v}$ to mean $u_i \le v_i$ for all i and $\mathbf{u} < \mathbf{v}$ to mean $u_i < v_i$ for all i.

Proposition 2.6. *Let $I \subseteq S$ be a monomial ideal. Then with the notation above:*

$$I^{[\frac{k}{q}]} = (\mathbf{x}^{\lfloor \frac{A\mathbf{u}}{q} \rfloor} \mid \mathbf{u} \in \mathbb{N}^n, \|\mathbf{u}\| = k, \tbinom{k}{\mathbf{u}} \not\equiv 0 \pmod{p})$$

Proof. By definition, $I^{[\frac{k}{q}]} = (I^{[k]})^{[\frac{1}{q}]}$ where

$$I^{[k]} = (\mathbf{x}^{A\mathbf{u}} \mid \mathbf{u} \in \mathbb{N}^n, \|\mathbf{u}\| = k, \tbinom{k}{\mathbf{u}} \not\equiv 0 \pmod{p})$$

by [HTW18, 3.5]. Since the monomials $\mathbf{x}^{\mathbf{b}}$ with $\mathbf{b} < q\mathbf{1}$ form a basis for S as a free S^q-module, we can compute the q-th root Frobenius power of an ideal $J = (f_1, \dots, f_n) \subseteq S$ as $J^{[\frac{1}{q}]} = (f_{i,\mathbf{b}} \mid f = \sum_{\mathbf{b} < q\mathbf{1}} f_{i,\mathbf{b}}^q \mathbf{x}^{\mathbf{b}})$ by [BMS08, 2.5]. Applying this description to $I^{[k]}$ yields the claimed description for $I^{[\frac{k}{q}]}$. □

Since every Frobenius power $I^{[\lambda]}$ agrees with the Frobenius power of some p-adic rational exponent, we have the following as an immediate consequence.

Corollary 2.7. *The Frobenius powers $I^{[\lambda]}$ of a monomial ideal $I \subseteq S$ are monomial ideals.*

Corollary 2.8. *Let $I \subseteq S$ be a monomial ideal as above. If x_j^2 does not divide any $\mathbf{x}^{\mathbf{a}_i}$, then for every $0 \le \lambda < 1$, x_j does not divide any generator of $I^{[\lambda]}$.*

Proof. Without loss of generality, we may assume $\lambda = \frac{k}{q}$ is a p-adic rational. By the above proposition, the generators of $I^{[\frac{k}{q}]}$ have the form $\mathbf{x}^{\lfloor \frac{A\mathbf{u}}{q} \rfloor}$, for some vector $\mathbf{u} \in \mathbb{N}^n$ such that $\|\mathbf{u}\| = k$. Since each $\mathbf{x}^{\mathbf{a}_i}$ is not divisible by x_j^2, the j-th row of A

contains no entries greater than one, and we have $(A\mathbf{u})_j \leq \|\mathbf{u}\| = k < q$. Hence, the exponent of x_j is $\lfloor \frac{(A\mathbf{u})_j}{q} \rfloor = 0$. $\square$

Corollary 2.9. *If $I \subseteq S$ is a monomial ideal that contains a squarefree monomial, then* $\operatorname{lce}(I) = 1$.

Proof. If $\mathbf{x}^{\mathbf{a}_1}$ is a squarefree monomial generator of I and $\mathbf{e}_1 \in \mathbb{N}^n$ denotes the corresponding standard basis vector, then $I^{[\frac{k}{q}]}$ contains $\mathbf{x}^{\lfloor \frac{A(k\mathbf{e}_1)}{q} \rfloor} = \mathbf{x}^{\lfloor \frac{k\mathbf{a}_1}{q} \rfloor} = 1$. Hence, $I^{[\frac{k}{q}]} = S$ for all $\frac{k}{q} < 1$. $\square$

As a consequence of the above corollary, every squarefree monomial ideal has least critical exponent equal to one. This is not surprising as the least critical exponent is supposed to be an analog of the F-pure threshold, and squarefree monomial ideals are F-pure. However, a monomial ideal need not be F-pure in order to have least critical exponent equal to one.

Example 2.10. The ideal $I = (x^2, xy) \subseteq S = \mathbb{F}_p[x, y]$ is not F-pure by Fedder's criterion [Fed83, 1.12] since

$$(I^{[p]} : I) = (x^{2p-2}, x^{p-2}y^p) \cap (x^{2p-1}, x^{p-1}y^{p-1}) = (x^{2p-1}, x^{2p-2}y^{p-1}, x^{p-1}y^p) \subseteq \mathfrak{m}^{[p]}$$

However, $\operatorname{lce}(I) = 1$ because I contains a square-free monomial.

Definition 2.11. For any monomial $\mathbf{x}^{\mathbf{b}} \in S$, we define the *critical exponent of* $\mathbf{x}^{\mathbf{b}}$ as:

$$\lambda_{\mathbf{b}}(I) = \sup\{\lambda \in \mathbb{R}_{\geq 0} : \mathbf{x}^{\mathbf{b}} \in I^{[\lambda]}\} = \sup\{\tfrac{k}{q} \in \mathbb{Z}[\tfrac{1}{p}]_{\geq 0} : \mathbf{x}^{\mathbf{b}} \in I^{[\frac{k}{q}]}\}$$

Since $I^{[k]} \subseteq I^k$ is generated in degrees at least k, it is clear that $\mathbf{x}^{\mathbf{b}} \notin I^{[k]}$ for $k = \|\mathbf{b}\| + 1$ so that the above suprema are always finite. That the two suprema are equal is clear from the right constancy of Frobenius powers and the density of p-adic rationals.

Remark 2.12. We note that $\mathbf{x}^{\mathbf{b}} \notin I^{[\lambda_{\mathbf{b}}]}$ by the right constancy of Frobenius powers so that $\lambda_{\mathbf{b}}(I)$ is in fact a critical exponent of I. By [HTW20, 2.5], the above definition coincides with what in that paper is called

$$\operatorname{crit}(I, \mathbf{b} + \mathbf{1}) = \sup\{\lambda \in \mathbb{R}_{\geq 0} : I^{[\lambda]} \not\subseteq (x_1^{b_1+1}, \ldots, x_m^{b_m+1})\}$$

In particular, we note that

$$\operatorname{lce}(I) = \lambda_{\mathbf{0}}(I) = \operatorname{crit}(I, \mathbf{1}) = \sup\{\lambda \in \mathbb{R}_{\geq 0} : I^{[\lambda]} \not\subseteq (x_1, \ldots, x_m)\}$$

and since $I^{[1]} = I \subseteq (x_1, \ldots, x_m)$, it follows that $0 < \operatorname{lce}(I) \leq 1$.

Proposition 2.13. *Every critical exponent of a monomial ideal $I \subseteq S$ is of the form $\lambda_{\mathbf{b}}(I)$ for some $\mathbf{b} \in \mathbb{N}^n$.*

Proof. Although we do not assume that I is $\mathfrak{m}$-primary, the same argument as in [HTW20, 2.6] shows that every critical exponent of I is of the form $\operatorname{crit}(I, \mathbf{a})$, except that a priori we only have $\mathbf{a} \in \mathbb{N}^n$ with $\mathbf{a} \neq \mathbf{0}$ instead of $\mathbf{a} > \mathbf{0}$. However, since we know that the Frobenius power $I^{[\lambda]}$ is a monomial ideal by Corollary 2.7, it is easily seen that $I^{[\lambda]} \not\subseteq (x_1^{a_1}, \dots, x_n^{a_n})$ if and only if the monomial $m_{\mathbf{a}} = \prod_{a_i > 0} x^{a_i - 1}$ is contained in $I^{[\lambda]}$. Taking $\tilde{\mathbf{a}} = \max(\mathbf{a}, \mathbf{1})$, it is clear that $m_{\tilde{\mathbf{a}}} = m_{\mathbf{a}}$ so that $I^{[\lambda]} \not\subseteq (x_1^{a_1}, \dots, x_n^{a_n})$ if and only if $I^{[\lambda]} \not\subseteq (x_1^{\tilde{a}_1}, \dots, x_n^{\tilde{a}_n})$. Hence, we have $\operatorname{crit}(I, \mathbf{a}) = \operatorname{crit}(I, \tilde{\mathbf{a}}) = \lambda_{\mathbf{b}}(I)$ for $\mathbf{b} = \tilde{\mathbf{a}} - \mathbf{1}$. $\square$

Remark 2.14. Due to their close relationship with test ideals of principal ideals, Skoda's Theorem for Frobenius powers [HTW18, 3.17] implies that every critical exponent $\lambda > 0$ satisfies that $\lambda - \lceil \lambda \rceil + 1$ is also a critical exponent in the interval $(0, 1]$. Consequently, we will only concern ourselves with finding critical exponents in this interval.

2.3. **Addition Base p and the Sierpinski Simplex.** Our next task is to shed some light on the condition $\binom{k}{\mathbf{u}} \not\equiv 0 \pmod{p}$. There is a useful interpretation in terms of the base p expansion of $\mathbf{u}$.

Lemma 2.15. *Suppose $\|\mathbf{u}\| = k$. We have $\binom{k}{\mathbf{u}} \not\equiv 0 \pmod{p}$ if and only if the addition $\sum u_i = k$ involves no carries in base p. That is, if we write $\mathbf{u} = (u_1, \dots, u_n)$ and write each $u_i = \sum_j u_{i,j} p^j$, then for all j we have $\sum_i u_{i,j} < p$.*

Proof. For an integer m, let $\nu_p(m)$ represent the number of times m is divisible by p. Then, if the base p expansion of m is $m = \sum c_i p^i$, we have $\nu_p(m!) = \sum_{i \neq 0} c_i (1 + p + p^2 + \cdots + p^{i-1})$.

Now observe that $\nu_p \binom{k}{\mathbf{u}} = \nu_p(k!) - \sum \nu_p(u_i!)$. Writing $k = \sum a_j p^j$ in base p, we have $a_j = \sum v_{i,j} + c_{j-1} - p c_j$, where c_j is the number of carries in the p^j place. We compute $\nu_p \binom{k}{\mathbf{u}} = \sum c_j$. Thus, $\binom{k}{\mathbf{u}} \equiv 0 \pmod{p}$ if and only if $\nu_p \binom{k}{\mathbf{u}} \geq 1$ if and only if there are carries in the addition. $\square$

The description in terms of addition base p allows us to translate from vectors of integers $\mathbf{u}$ with $\|\mathbf{u}\| = k$ to vectors of p-adic rational numbers $\mathbf{v}$ with $\|\mathbf{v}\| = \frac{k}{q}$.

Definition 2.16. Fix a prime p and a dimension n. The (p, n)-*Sierpinski simplex* is the set $\mathcal{S}_{p,n}$ consisting of all n-tuples $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ such that each v_i is a terminating decimal in base p and these decimals add without a carry. The *closed* (p, n)-*Sierpinski simplex* is the set $\overline{\mathcal{S}}_{p,n}$ consisting of all real admissible n-tuples $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ for which it is possible to choose base p representations of every entry v_i in such a way that $\sum v_i$ adds without carries.

Remark 2.17. The distinction between the open and closed Sierpinski simplices isnot simply the distinction between p-adic fractions and real numbers. The non-uniqueness of decimal representations for terminating decimals allows the closed Sierpinski simplex to contain many rational points that are missing from the open simplex. For example, $\mathcal{S}_{2,2}$ does not contain $(\frac{1}{2}, \frac{1}{2}) = (.1, .1)$ because the sum

$.1 + .1 = 1$ involves a carry. However, $\overline{\mathcal{S}}_{2,2}$ does contain this point, because we may choose to write it as $(.1, .0\overline{1})$, and the sum $.1 + .0\overline{1} = .\overline{1}\ldots$ does not require a carry.

Remark 2.18. The sets $\mathcal{S}_{p,n}$ and $\overline{\mathcal{S}}_{p,n}$ are fractals. $\overline{\mathcal{S}}_{2,2}$ is the familiar Sierpinski gasket, and $\overline{\mathcal{S}}_{p,n}$ has dimension $\log_p \binom{p+n-1}{n}$. We provide several iterative methods for building these fractals.

Method one: Set $X = \{\mathbf{v} \in \mathbb{N}^n : \|\mathbf{v}\| < p\}$. Put $S_1 = \{\frac{1}{p}\mathbf{v} : \mathbf{v} \in X\}$, $S_2 = \{\mathbf{u} + \frac{1}{p^2}\mathbf{v} : \mathbf{u} \in S_1, \mathbf{v} \in X\}$, and in general $S_i = \{\mathbf{u} + \frac{1}{p^i}\mathbf{v} : \mathbf{u} \in S_{i-1}, \mathbf{v} \in X\}$. Then $\mathcal{S}_{p,n} = \bigcup S_i$. Equivalently, $\mathcal{S}_{p,n} = X + \frac{1}{p}\mathcal{S}_{p,n}$.

Method two: Let T_0 be the unit hypercube $\{(x_1, \ldots, x_n) : 0 \leq x_i \leq 1\}$. Subdivide T_0 into p^n congruent hypercubes of side length $\frac{1}{p}$ in the standard way. Then delete all the smaller cubes that lie entirely in the half-space $\sum x_i \geq 1$. Call the result T_1. Then replace each of the cubes in T_1 with a $\frac{1}{p}$-scale copy of T_1; the result is T_2. In general, obtain T_i by replacing each of the surviving cubes in T_1 with a $\frac{1}{p^{i-1}}$-scale copy of T_{i-1} (or, equivalently, replace each of the surviving cubes in T_{i-1} with a copy of T_1). Then $\overline{\mathcal{S}}_{p,n} = \cap T_i$.

Method three: Let W_0 be the unit n-simplex (the convex hull of the origin and the n standard basis vectors). Divide each of the edges into p equal segments, and draw in all hyperplanes parallel to the sides of W_0 and through the new vertices. This divides W_0 into $\binom{p+n-1}{n} + \binom{p+n-2}{n}$ congruent subsimplices, of which $\binom{p+n-1}{n}$ are oriented correctly. Delete the backwards simplices and call the result W_1. In general, obtain W_{i+1} by replacing each simplex in W_{i+1} with a scaled-down copy of W_1.

Theorem 2.19. *Let I be a monomial ideal and $\mathbf{b} \in \mathbb{N}^m$. Then*

$$\lambda_{\mathbf{b}}(I) = \sup\{\|\mathbf{v}\| : \mathbf{v} \in \mathcal{S}_{p,n}, A\mathbf{v} < \mathbf{b} + \mathbf{1}\}.$$

Proof. If $\mathbf{v} \in \mathcal{S}_{p,n}$ with $A\mathbf{v} < \mathbf{b} + \mathbf{1}$, then $\|\mathbf{v}\|$ is a p-adic rational, say $\|\mathbf{v}\| = \frac{k}{q}$, and we can write $\mathbf{v} = \frac{\mathbf{u}}{q}$ for some $\mathbf{u} \in \mathbb{N}^n$ with $\|\mathbf{u}\| = k$. Since $\mathbf{v} \in \mathcal{S}_{p,n}$, it follows from Lemma 2.15 that $\binom{k}{\mathbf{u}} \not\equiv 0 \pmod{p}$ so that $\mathbf{x}^{\lfloor \frac{A\mathbf{u}}{q} \rfloor} \in I^{[\frac{k}{q}]}$ by Proposition 2.7. As $\lfloor \frac{A\mathbf{u}}{q} \rfloor = \lfloor A\mathbf{v} \rfloor \leq \mathbf{b}$, it follows that $\mathbf{x}^{\mathbf{b}} \in I^{[\frac{k}{q}]}$. Hence, $\frac{k}{q} = \|\mathbf{v}\| \leq \lambda_{\mathbf{b}}(I)$, which shows that the supremum on the right in statement of the theorem is at most $\lambda_{\mathbf{b}}(I)$. The reverse inequality is proved similarly. $\square$

Remark 2.20. The supremum in the above theorem is *never* attained by any $\mathbf{v} \in \mathcal{S}_{p,n}$ with $A\mathbf{v} < \mathbf{b} + \mathbf{1}$; if it were, we would have $\lambda = \lambda_{\mathbf{b}}(I) = \|\mathbf{v}\|$ and $\mathbf{x}^{\mathbf{b}} \in I^{[\lambda]}$ as the proof of the theorem shows. But then $\mathbf{x}^{\mathbf{b}} \in I^{[\lambda+\varepsilon]}$ for some $\varepsilon > 0$ by the right constancy of Frobenius powers (Proposition 2.1), contradicting the definition of $\lambda_{\mathbf{b}}(I)$.

2.4. **Truncations and Witnesses.** Partially following [Her12], we introduce notation for truncating the base p expansions of real numbers.

Definition 2.21. Let $e \geq 0$ be an integer and $\alpha \in (0,1]$ with nonterminating base p expansion $\alpha = \sum_{i=1}^{\infty} \frac{\alpha^{(i)}}{p^i}$.

(i) The *e-th truncation* of α is $\langle \alpha \rangle_e = \frac{\alpha^{(1)}}{p} + \frac{\alpha^{(2)}}{p^2} + \cdots + \frac{\alpha^{(e)}}{p^e}$ if $e \geq 1$, and $\langle \alpha \rangle_0 = 0$.

(ii) If $\alpha \in \mathbb{Z}[\frac{1}{p}]$ with terminating base p expansion $\alpha = \frac{\alpha_1}{p} + \frac{\alpha_2}{p^2} + \cdots + \frac{\alpha_r}{p^r}$ where $\alpha_r \neq 0$, we define the *e-th strict truncation* of α by

$$\sigma_e(\alpha) = \begin{cases} \alpha & \text{if } e > r \\ \frac{\alpha_1}{p} + \frac{\alpha_2}{p^2} + \cdots + \frac{\alpha_e}{p^e} & \text{if } e \leq r \end{cases}$$

We also set $\langle 0 \rangle_e = \sigma_e(0) = 0$ for all e.

Lemma 2.22 ([Her12, 2.5])**.** *Let $\alpha, \beta \in [0,1]$ and $e \geq 0$ be an integer.*

(a) *If $\alpha > 0$, then $\lceil p^e \alpha \rceil = p^e \langle \alpha \rangle_e + 1$.*
(b) *If $\beta \in \frac{1}{p^e}\mathbb{N}$ and $\beta < \alpha$, then $\beta \leq \langle \alpha \rangle_e$.*
(c) *If $\beta \leq \alpha$, then $\langle \beta \rangle_e \leq \langle \alpha \rangle_e$.*

As usual, we also extend the preceding definitions to vectors componentwise to make the following definition.

Definition 2.23. Given an integer $e \geq 0$, an *e-witness* for $\lambda = \lambda_{\mathbf{b}} \in (0,1]$ is a vector $\mathbf{w} \in \mathcal{S}_{p,n}$ such that $A\mathbf{w} < \mathbf{b} + \mathbf{1}$ and $\|\mathbf{w}\| = \langle \lambda \rangle_e$. We denote the set of all e-witnesses for λ by $\mathcal{W}_e$. A *family of witnesses* for λ is a sequence of vectors $\{\mathbf{w}_e\}$ such that $\mathbf{w}_e$ is an e-witness for λ and $\sigma_e(\mathbf{w}_{e+1}) = \mathbf{w}_e$ for all e.

Remark 2.24. We record some simple observations about witness vectors that will be useful below:

- If $\mathbf{w}$ is an e-witness for λ, then $\mathbf{w} \in \frac{1}{p^e}\mathbb{N}^n$ since $\mathbf{w} \in \mathcal{S}_{p,n}$ and $\|\mathbf{w}\| = \langle \lambda \rangle_e \in \frac{1}{p^e}\mathbb{N}$. In particular, a 0-witness is a vector $\mathbf{w} \in \mathbb{N}^n$ with $\|\mathbf{w}\| = \langle \lambda \rangle_0 = 0$ so that $\mathcal{W}_0 = \{\mathbf{0}\}$.
- If $\mathbf{w}$ is an e'-witness for λ and $e \leq e'$, then $\sigma_e(\mathbf{w})$ is easily seen to be an e-witness for λ since $\|\sigma_e(\mathbf{w})\| = \sigma_e(\|\mathbf{w}\|) = \sigma_e(\langle \lambda \rangle_{e'}) = \langle \lambda \rangle_e$. Moreover, we have $\|\mathbf{w} - \sigma_e(\mathbf{w})\| = \|\mathbf{w}\| - \|\sigma_e(\mathbf{w})\| = \langle \lambda \rangle_{e'} - \langle \lambda \rangle_e \leq \frac{p-1}{p^{e+1}} + \cdots + \frac{p-1}{p^{e'}} = p^{-e} - p^{-e'}$.

Theorem 2.25. *Let $\lambda = \lambda_{\mathbf{b}}$ be a critical exponent of I with $\lambda \in (0,1]$.*

(a) *If $\{\mathbf{w}_e\}$ is a family of witnesses for λ, then $\mathbf{w} = \lim_{e\to\infty} \mathbf{w}_e$ exists, and $\|\mathbf{w}\| = \lambda$.*
(b) *For every integer $e \geq 0$, there exists an e-witness for λ.*
(c) *There exists a family of witnesses for λ.*

Proof. (a) If $e \leq e'$, then $\sigma_e(\mathbf{w}_{e'}) = \mathbf{w}_e$ since $\{\mathbf{w}_e\}$ is a family of witnesses. Hence, Remark 2.24 shows that $\|\mathbf{w}_{e'} - \mathbf{w}_e\| < p^{-e}$ for all $e' \geq e$ so that $\{\mathbf{w}_e\}$ is a Cauchy sequence and, therefore, converges. Consequently, $\|\mathbf{w}\| = \lim_{e\to\infty} \|\mathbf{w}_e\| = \lim_{e\to\infty} \langle\lambda\rangle_e = \lambda$.

(b) First, note that $\langle\lambda\rangle_e < \lambda$ by definition since we choose the nonterminating representation of λ. By Theorem 2.19, we know there is a $\mathbf{w} \in \mathcal{S}_{p,n}$ with $A\mathbf{w} < \mathbf{b}+\mathbf{1}$ such that $\langle\lambda\rangle_e < \|\mathbf{w}\| \leq \lambda$. It then follows from Lemma 2.22 that $\langle\|\mathbf{w}\|\rangle_e = \langle\lambda\rangle_e$. We also note that $\|\mathbf{w}\| \notin \frac{1}{p^e}\mathbb{N}$, since otherwise we would have $\|\mathbf{w}\| \leq \langle\lambda\rangle_e$ by Lemma 2.22. Hence, the terminating base p expansion of $\|\mathbf{w}\|$ has more than e decimals so that $\sigma_e(\|\mathbf{w}\|) = \langle\|\mathbf{w}\|\rangle_e$. Then $\sigma_e(\mathbf{w}) \in \mathcal{S}_{p,n}$ since $\mathbf{w} \in \mathcal{S}_{p,n}$, and $A\sigma_e(\mathbf{w}) \leq A\mathbf{w} < \mathbf{b}+\mathbf{1}$. As $\|\sigma_e(\mathbf{w})\| = \sigma_e(\|\mathbf{w}\|) = \langle\|\mathbf{w}\|\rangle_e = \langle\lambda\rangle_e$, it follows that $\sigma_e(\mathbf{w})$ is an e-witness for λ.

(c) By the previous part, there exists a sequence of vectors $\{\mathbf{w}_e\}$ in the set $P = \{\mathbf{w} \in [0,1]^n : A\mathbf{w} \leq \mathbf{b}+\mathbf{1}\}$ such that $\mathbf{w}_e$ is an e-witness for every e. Since this set is compact, there is a subsequence $\{\mathbf{w}_{e_k}\}$ that converges to some vector $\mathbf{w}$.

We will now recursively construct for each $e \geq 0$ a sequence of vectors $\{\mathbf{w}_k^{(e)}\}$ and vectors $\mathbf{v}_0, \ldots, \mathbf{v}_e$ such that $\mathbf{w}_k^{(e)} \to \mathbf{w}$, $\mathbf{w}_k^{(e)}$ is an e_k-witness for some $e_k \geq k$ for each k, $\mathbf{v}_h$ is an h-witness for $0 \leq h \leq e$, and $\sigma_h(\mathbf{w}_k^{(e)}) = \mathbf{v}_h$ for all $k \gg 0$ for each h. The resulting sequence $\{\mathbf{v}_e\}$ will be the desired family of witnesses, which also converges to $\mathbf{w}$. Set $\mathbf{w}_k^{(0)} = \mathbf{w}_{e_k}$, the terms of the convergent subsequence from the previous paragraph. Taking $\mathbf{v}_0 = \mathbf{0}$, it is clear that $\sigma_0(\mathbf{w}_k^{(0)}) = \mathbf{v}_0$ for all k so that $\{\mathbf{w}_k^{(0)}\}$ has the desired properties.

Suppose that we have already constructed the sequence $\{\mathbf{w}_k^{(e)}\}$ and the vectors $\mathbf{v}_0, \ldots, \mathbf{v}_e$. Now, consider the sequence of truncations $\{\sigma_{e+1}(\mathbf{w}_k^{(e)})\}$. Since each truncation is still contained in the compact set P, this sequence has a convergent subsequence $\sigma_{e+1}(\mathbf{w}_{k_\ell}^{(e)}) \to \mathbf{v}_{e+1}$. However, since each truncation is an $(e+1)$-witness (for at least $k \geq e+1$), every term of this subsequence is eventually contained in the discrete set $\frac{1}{p^{e+1}}\mathbb{N}^n$. Hence, this subsequence is eventually constant so that $\sigma_{e+1}(\mathbf{w}_{k_\ell}^{(e)}) = \mathbf{v}_{e+1}$ for all $\ell \gg 0$. In particular, we see that $\mathbf{v}_{e+1}$ is an $(e+1)$-witness. Hence, if we define $\mathbf{w}_\ell^{(e+1)} = \mathbf{w}_{k_\ell}^{(e)}$, then $\mathbf{w}_\ell^{(e+1)} \to \mathbf{w}$ since it is subsequence of a sequence converging to $\mathbf{w}$, $\mathbf{w}_\ell^{(e+1)}$ is an e_ℓ-witness for some $e_\ell = e_{k_\ell} \geq k_\ell \geq \ell$, and $\sigma_h(\mathbf{w}_\ell^{(e+1)}) = \sigma_h(\mathbf{w}_{k_\ell}^{(e)}) = \mathbf{v}_h$ for all $\ell \gg 0$ for $0 \leq h \leq e+1$. Thus, we can construct the desired sequences of vectors recursively.

To see that the resulting sequence of e-witnesses $\{\mathbf{v}_e\}$ is a family of witnesses, we note that $\sigma_e(\mathbf{v}_{e+1}) = \sigma_e(\sigma_{e+1}(\mathbf{w}_k^{(e+1)})) = \sigma_e(\mathbf{w}_k^{(e+1)}) = \mathbf{v}_e$ for all e and $k \gg 0$. ☐

Remark 2.26. The argument of recursively passing to subsequences in the proof of part (c) of the theorem above seems somewhat necessary. For example, suppose that $p = 5$, and consider the sequence $\{\mathbf{w}_e\}_{e\geq 2}$ defined by $\mathbf{w}_e = (\frac{2}{5}, \frac{2}{5} - \frac{1}{5^e}) =$

$(0.2, 0.144\ldots 4)$ if e is even and $\mathbf{w}_e = (\frac{2}{5} - \frac{1}{5^e}, \frac{2}{5}) = (0.144\ldots 4, 0.2)$ if e is odd. Clearly, we have $\mathbf{w}_e \to \mathbf{w} = (\frac{2}{5}, \frac{2}{5})$. However, for $k \geq e$, the sequence of truncations $\{\sigma_e(\mathbf{w}_k)\}$ is given by $\sigma_e(\mathbf{w}_k) = (\frac{2}{5}, \frac{2}{5} - \frac{1}{5^e})$ if k is even and $\sigma_e(\mathbf{w}_k) = (\frac{2}{5} - \frac{1}{5^e}, \frac{2}{5})$ if k is odd. Thus, we must pass to a subsequence (whose terms will have indices that are eventually always even or always odd) in order to obtain a single stable limit vector. Moreover, we must do this recursively to ensure that we are not picking $\mathbf{v}_e = (\frac{2}{5}, \frac{2}{5} - \frac{1}{5^e})$ but then $\mathbf{v}_{e+1} = (\frac{2}{5} - \frac{1}{5^{e+1}}, \frac{2}{5})$ for example.

3. An Algorithm for Computing Critical Exponents

Our strategy for computing the critical exponent $\lambda_{\mathbf{b}}(I)$ associated to $\mathbf{b} \in \mathbb{N}^m$ is to recursively compute its base p expansion to e decimal places of accuracy. We begin by defining an (infinite) process that computes all e-witnesses to $\lambda_{\mathbf{b}}$ for all e. Later in Algorithm 3.7, we show this can be adapted to a terminating algorithm.

Proposition 3.1. *Let $\lambda = \lambda_{\mathbf{b}}$ be a critical exponent of I with $\lambda \in (0, 1]$. If $\mathbf{w} \in \mathcal{W}_e$ for some $e \geq 1$ and we write $\mathbf{w} = \sigma_{e-1}(\mathbf{w}) + \frac{\mathbf{z}}{p^e}$ for some $\mathbf{z} \in \mathbb{N}^n$, then $\sigma_{e-1}(\mathbf{w}) \in \mathcal{W}_{e-1}$ and*

$$\|\mathbf{z}\| = \max\{\|\mathbf{v}\| : \mathbf{v} \in \mathbb{N}^n, A\mathbf{v} < p^e\mathbf{r}, \|\mathbf{v}\| < p\}$$

where $\mathbf{r} = \mathbf{b} + \mathbf{1} - A\sigma_{e-1}(\mathbf{w})$.

Proof. We have already observed that $\sigma_{e-1}(\mathbf{w}) \in \mathcal{W}_{e-1}$ in Remark 2.24. Let m be the maximum on the righthand side above. It is clear that $A\mathbf{w} < \mathbf{b} + \mathbf{1}$ implies $A\mathbf{z} < p^e\mathbf{r}$. Since $\mathbf{w} \in \mathcal{S}_{p,n}$ and $\sigma_{e-1}(\mathbf{w}) \in \frac{1}{p^{e-1}}\mathbb{N}^n$, we must also have $\|\mathbf{z}\| < p$ so that $\|\mathbf{z}\| \leq m$. If $\mathbf{z} < m$, then there is a vector $\mathbf{v} \in \mathbb{N}^n$ with $A\mathbf{v} < p^e\mathbf{r}$ and $\|\mathbf{z}\| < \|\mathbf{v}\| < p$. Then $\mathbf{w}' = \sigma_{e-1}(\mathbf{w}) + \frac{\mathbf{v}}{p^e} \in \mathcal{S}_{p,n}$ and $A\mathbf{w}' < \mathbf{b} + \mathbf{1}$ so that $\|\mathbf{w}'\| < \lambda$ by Theorem 2.19 and Remark 2.20. Consequently, Lemma 2.22 implies $\|\sigma_{e-1}(\mathbf{w})\| + \frac{\|\mathbf{v}\|}{p^e} = \|\mathbf{w}'\| \leq \langle\lambda\rangle_e = \|\mathbf{w}\| = \|\sigma_{e-1}(\mathbf{w})\| + \frac{\|\mathbf{z}\|}{p^e}$, contradicting that $\|\mathbf{z}\| < \|\mathbf{v}\|$. Hence, $\|\mathbf{z}\| = m$ as claimed. $\square$

Algorithm 3.2. Fix a monomial $\mathbf{x}^{\mathbf{b}} \notin I$, and let $e \geq 0$ an integer. Starting with $\mathcal{L}_0 = \{\mathbf{0}\}$, inductively compute the set $\mathcal{L}_e$ for each $e \geq 1$ as follows:

(1) For each $\mathbf{u} \in \mathcal{L}_{e-1}$, compute the remainder vector $\mathbf{r} = \mathbf{b} + \mathbf{1} - A\mathbf{u}$.
(2) Find all solutions $\mathbf{v} \in \mathbb{N}^n$ maximizing $\|\mathbf{v}\|$ subject to the constraints that $A\mathbf{v} < p^e\mathbf{r}$ and $\|\mathbf{v}\| < p$.
(3) Append each $\mathbf{u} + \frac{1}{p^e}\mathbf{v}$ to $\mathcal{L}_e$.
(4) After doing this for all $\mathbf{u}$, compute $\lambda_e = \max\{\|\mathbf{w}\| : \mathbf{w} \in \mathcal{L}_e\}$, and remove from $\mathcal{L}_e$ every $\mathbf{w}$ with $\|\mathbf{w}\| < \lambda_e$.

Remark 3.3. We make some simple observations about the preceding algorithm:

- The computation of all $\mathbf{v}$ in Step 2 terminates because there are finitely many non-negative integer vectors $\mathbf{v}$ with $\|\mathbf{v}\| < p$.

- Every element $\mathbf{w} = \mathbf{u} + \frac{1}{p^e}\mathbf{v} \in \mathcal{L}_e$ is contained in $\frac{1}{p^e}\mathbb{N}^n \cap \mathcal{S}_{p,n}$ since $\mathbf{u} \in \mathcal{L}_{e-1} \subseteq \frac{1}{p^{e-1}}\mathbb{N}^n \cap \mathcal{S}_{p,n}$ by induction and $\|\mathbf{v}\| < p$ so that computing the norm of $\mathbf{u} + \frac{1}{p^e}\mathbf{v}$ doesn't involve a carry.

Theorem 3.4. *For every integer $e \geq 0$, the set of vectors $\mathcal{L}_e$ produced by Algorithm 3.2 is precisely the set $\mathcal{W}_e$ of all e-witnesses for λ.*

Proof. We argue by induction on e. By Remark 2.24, it follows that $\mathcal{W}_0 = \mathcal{L}_0$. Suppose that $e \geq 1$ and $\mathcal{W}_{e-1} = \mathcal{L}_{e-1}$. If $\mathbf{w} \in \mathcal{W}_e$ for some $e \geq 1$ and we write $\mathbf{w} = \sigma_{e-1}(\mathbf{w}) + \frac{\mathbf{z}}{p^e}$ for some $\mathbf{z} \in \mathbb{N}^n$, it follows from Proposition 3.1 that $\sigma_{e-1}(\mathbf{w}) \in \mathcal{W}_{e-1} = \mathcal{L}_{e-1}$ and that $\mathbf{w}$ added to $\mathcal{L}_e$ by Step 3 of the algorithm. If it were the case that $\|\mathbf{w}\| < \lambda_e$ in Step 4 of the algorithm, then there would be a $\mathbf{u} \in \mathcal{L}_{e-1} = \mathcal{W}_{e-1}$ and $\mathbf{v} \in \mathbb{N}^n$ a solution from Step 2 of the algorithm such that $\left\|\mathbf{u} + \frac{\mathbf{v}}{p^e}\right\| > \|\mathbf{w}\| = \langle\lambda\rangle_e$. However, it also follows from Remark **??** that $\mathbf{u} + \frac{\mathbf{v}}{p^e} \in \frac{1}{p^e}\mathbb{N}^n \cap \mathcal{S}_{p,n}$, and $A(\mathbf{u} + \frac{\mathbf{v}}{p^e}) < \mathbf{b} + \mathbf{1}$ by construction. Hence, we have $\left\|\mathbf{u} + \frac{\mathbf{v}}{p^e}\right\| < \lambda$ by Theorem 2.19 and Remark 2.20 so that $\left\|\mathbf{u} + \frac{\mathbf{v}}{p^e}\right\| \leq \langle\lambda\rangle_e$ by Lemma 2.22, which is a contradiction. Thus, $\lambda_e = \|\mathbf{w}\| = \langle\lambda\rangle_e$, and $\mathbf{w}$ is not removed from $\mathcal{L}_e$ in Step 4 so that $\mathcal{W}_e \subseteq \mathcal{L}_e$. Conversely, if $\mathbf{w} \in \mathcal{L}_e$, then as already noted above we have $\mathbf{w} \in \mathcal{S}_{p,n}$ and $A\mathbf{w} < \mathbf{b} + \mathbf{1}$ by construction, and $\|\mathbf{w}\| = \lambda_e = \langle\lambda\rangle_e$ by Step 4 of the algorithm so that $\mathbf{w} \in \mathcal{W}_e$. Therefore, $\mathcal{L}_e = \mathcal{W}_e$ as claimed. □

In order to turn Algorithm 3.2 into an algorithm to compute $\lambda_{\mathbf{b}}$ with finite resources, we look for inspiration from the familiar algorithm using long division to produce decimal expansions of fractions. We recall a simple example of that algorithm below as motivation.

Example 3.5. We use long division to find the decimal expansion of $\frac{1}{22}$. Long division first divides 22 into 1, computing a quotient of 0 and a remainder of 1. It then multiplies the remainder by 10 and divides by 22 again, producing a quotient of 0 and a remainder of 10. At the next step, we get a quotient of 4 and a remainder of 12, followed by a quotient of 5 and a remainder of 10. At this point, the algorithm recognizes that the new remainder has appeared before, so all steps from the previous remainder of 10 will repeat. Thus all future quotients will repeat in a pattern of 4, 5, 4, 5, ... We conclude that $\frac{1}{22} = 0.0\overline{45}$.

Unfortunately, adapting the idea of long division to our vector algorithm is slightly complicated because there may be multiple vectors $\mathbf{v}$ for a given $\mathbf{u}$, so our vectors of digits are not forced to repeat in the same pattern. As a means of circumventing these difficulties, we make the following observation.

Remark 3.6. With the notation of Algorithm 3.2, since the exponent matrix A is fixed, there exists an integer Ω such that

$$\max\{\|A\mathbf{v}\|_\infty : \mathbf{v} \in \mathbb{N}^n, \|\mathbf{v}\| < p\} \leq (1 - \tfrac{1}{p})\Omega \tag{3.1}$$

That is, every entry of the vectors $A\mathbf{v}$ is at most $(1 - \frac{1}{p})\Omega$. Consequently, if any entry of the vector $p^e\mathbf{r}$ is greater than Ω, we may replace that entry with Ω without changing the result of the algorithm. In particular, after this modification, there are only finitely many possible remainder vectors since $p^e\mathbf{r}$ always has nonnegative integer entries.

This motivates the following reformulation of Algorithm 3.2 as an algorithm that will terminate after finitely many steps.

Algorithm 3.7. Fix a monomial $\mathbf{x}^\mathbf{b} \notin I$ and an integer Ω as in Remark 3.6, and let $e \geq 0$ be an integer. Starting with $\mathcal{L}_0 = \{\mathbf{0}\}$ and $\mathcal{L}_\infty = \varnothing$, inductively compute the sets $\mathcal{L}_e$ and $\mathcal{L}_\infty$ for each $e \geq 0$ as follows:

(1) If $\mathcal{L}_e = \varnothing$, then output $\lambda = \max\{\|\mathbf{w}\| : \mathbf{w} \in \mathcal{L}_\infty\}$.
(2) Otherwise, for each $\mathbf{u} \in \mathcal{L}_e$, compute the remainder vector
$$\mathbf{r}(e, \mathbf{u}) := \min(p^{e+1}(\mathbf{b} + \mathbf{1} - A\mathbf{u}), \Omega\mathbf{1}).$$
(3) If $\mathbf{r}(e, \mathbf{u}) = \mathbf{r}(e - s, \sigma_{e-s}(\mathbf{u}))$ for some s with $e \geq s \geq 1$, then append the vector
$$\mathbf{u} + \frac{1}{p^s - 1}(\mathbf{u} - \sigma_{e-s}(\mathbf{u}))$$
to $\mathcal{L}_\infty$.
(4) Otherwise, find all solutions $\mathbf{v} \in \mathbb{N}^n$ maximizing $\|\mathbf{v}\|$ subject to the constraints that $A\mathbf{v} < \mathbf{r}(e, \mathbf{u})$ and $\|\mathbf{v}\| < p$, and append each $\mathbf{u} + \frac{1}{p^e}\mathbf{v}$ to $\mathcal{L}_{e+1}$.
(5) After doing this for all $\mathbf{u}$, if $\mathcal{L}_{e+1} \neq \varnothing$, compute $\lambda_{e+1} = \max\{\|\mathbf{w}\| : \mathbf{w} \in \mathcal{L}_{e+1}\}$, and remove from $\mathcal{L}_{e+1}$ every $\mathbf{w}$ with $\|\mathbf{w}\| < \lambda_{e+1}$.

Remark 3.8. Since there are finitely many possible remainder vectors by construction, $\mathcal{L}_e$ will be empty for sufficiently large e. At each stage, $\mathcal{L}_e$ is a finite set, so we can add only finitely many vectors to $\mathcal{L}_\infty$. Hence, $\mathcal{L}_\infty$ will be a finite set when the algorithm terminates.

Note also that both Algorithms 3.2 and 3.7 depend only on the exponent matrix A, so we need not be given a minimal set of generators for I. We will exploit this fact later in Example 5.4.

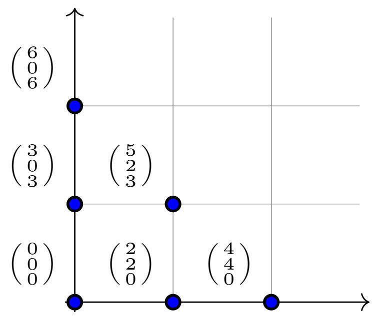
Example 3.9. Here, we use Algorithm 3.7 to compute the least critical exponent of the ideal $I = (x^2y^2, x^3z^3)$ in $S = \mathbb{F}_3[x, y, z]$. Thus, we have $p = 3$ and an exponent matrix

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 3 \end{pmatrix},$$

and we will compute $\lambda_{\mathbf{b}}$ where $\mathbf{b} = (0,0,0)$. To ensure that the remainder vectors eventually repeat, we must choose an integer Ω such that

$$\max\{\|A\mathbf{v}\|_\infty : \mathbf{v} \in \mathbb{N}^2, \|\mathbf{v}\| < p\} \leq (1 - \tfrac{1}{p})\Omega.$$

The vectors $\mathbf{v} \in \mathbb{N}^2$ with $\|\mathbf{v}\| < 3$ are shown below labeled by their values $A\mathbf{v}$.



From this, we easily see that the smallest possible value for Ω is $\Omega = 9$.

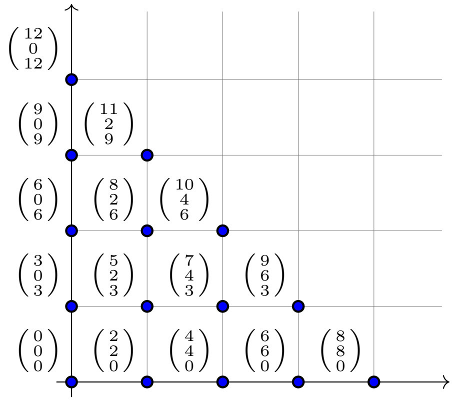
Table 1 below summarizes each step of the Algorithm 3.7 for this example. Each row shows the list of vectors currently in the sets $\mathcal{L}_\infty$ and $\mathcal{L}_e$ at each step as well as the corresponding remainder vectors $\mathbf{r}(e, \mathbf{u})$.

e	$\mathcal{L}_\infty$	$\mathcal{L}_e$	$\mathbf{r}(e,\mathbf{u})$
0	$\varnothing$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$
1	$\varnothing$	$\frac{1}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$
2	$\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\varnothing$	–

Table 1. Algorithm 3.7 for the least critical exponent of the ideal $(x^2y^2, y^3z^3) \subseteq \mathbb{F}_3[x,y,z]$

In this case, we note that $(1,0)$ is the only vector $\mathbf{v} \in \mathbb{N}^2$ with $\|\mathbf{v}\| < 3$ and $A\mathbf{v} < (3,3,3)$ of maximum norm. The remainder immediately repeats at the next step so that the appropriate limit vector is added to $\mathcal{L}_\infty$, and the algorithm then terminates since $\mathcal{L}_2$ is empty, outputting $\operatorname{lce}(I) = \frac{1}{2}$. This agrees with Proposition 4.1 below.

Example 3.10. We now demonstrate a more involved example. Consider the ideal $I = (x^2y^2, x^3z^3)$ in $S = \mathbb{F}_5[x, y, z]$ so that $p = 5$. We will compute $\lambda_{\mathbf{b}}$ for $\mathbf{b} = (1, 1, 0)$, the smallest Frobenius power of I excluding the monomial xy. The vectors $\mathbf{v} \in \mathbb{N}^2$ with $\|\mathbf{v}\| < 5$ are shown below labeled by their values $A\mathbf{v}$.



From this, we easily see that the smallest possible value for Ω is $\Omega = 15$.

e	$\mathcal{L}_\infty$	$\mathcal{L}_e$	$\mathbf{r}(e, \mathbf{u})$
0	$\varnothing$	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}10\\10\\5\end{pmatrix}$
1	$\varnothing$	$\frac{1}{5}\begin{pmatrix}4\\0\end{pmatrix}, \frac{1}{5}\begin{pmatrix}3\\1\end{pmatrix}$	$\begin{pmatrix}10\\10\\15\end{pmatrix}, \begin{pmatrix}5\\15\\10\end{pmatrix}$
2	$\varnothing$	$\frac{1}{5}\begin{pmatrix}4\\0\end{pmatrix} + \frac{1}{25}\begin{pmatrix}4\\0\end{pmatrix} = \frac{1}{25}\begin{pmatrix}24\\0\end{pmatrix},$ $\frac{1}{5}\begin{pmatrix}4\\0\end{pmatrix} + \frac{1}{25}\begin{pmatrix}3\\1\end{pmatrix} = \frac{1}{25}\begin{pmatrix}23\\1\end{pmatrix},$ $\frac{1}{5}\begin{pmatrix}3\\1\end{pmatrix} + \frac{1}{25}\begin{pmatrix}2\\0\end{pmatrix} = \frac{1}{25}\begin{pmatrix}17\\5\end{pmatrix}$ $(*)$	$\begin{pmatrix}10\\10\\15\end{pmatrix}, \begin{pmatrix}5\\15\\15\end{pmatrix}$
3	$\begin{pmatrix}1\\0\end{pmatrix}$	$\frac{1}{25}\begin{pmatrix}23\\1\end{pmatrix} + \frac{1}{125}\begin{pmatrix}2\\0\end{pmatrix} = \frac{1}{125}\begin{pmatrix}117\\5\end{pmatrix},$	$\begin{pmatrix}5\\15\\15\end{pmatrix}$
4	$\begin{pmatrix}1\\0\end{pmatrix}, \frac{1}{50}\begin{pmatrix}47\\2\end{pmatrix}$	$\varnothing$	–

TABLE 2. Algorithm 3.7 for a critical exponent of the ideal $(x^2y^2, y^3z^3) \subseteq \mathbb{F}_5[x, y, z]$

Table 2 above summarizes each step of the Algorithm 3.7 for this example. In the previous example, our work was made easier by the fact that there was only ever one $\mathbf{v}$ found in Step 2 of the algorithm. In general, however, there will likely be several such vectors found, and the magnitudes of the resulting vectors appended to $\mathcal{L}_{e+1}$ must be compared. The vector computed during the second iteration that is marked by $(*)$ is removed from $\mathcal{L}_2$ by Step 5 of the algorithm since its norm is smaller than the other two vectors.

This example also shows the necessity of comparing the norms of the limit vectors in $\mathcal{L}_\infty$ when the algorithm terminates. Only the largest norm is the critical exponent. So, in this case, we see that $\lambda_{\mathbf{b}}(I) = 1$ (compare with Example 4.2).

The remainder of this section is devoted to proving that Algorithm 3.7 actually returns the desired critical exponent of a monomial ideal. To simplify some notation in the following lemma, we define

$$\eta(e, \mathbf{u}) := p^{e+1}(\mathbf{b} + \mathbf{1} - A\mathbf{u}) \tag{3.2}$$

for any vector $\mathbf{u} \in \mathcal{S}_{p,n} \cap \frac{1}{p^e}\mathbb{N}^n$ so that $\mathbf{r}(e, \mathbf{u}) = \min(\eta(e, \mathbf{u}), \Omega\mathbf{1})$.

Lemma 3.11. *Let $\mathbf{u} \in \frac{1}{p^e}\mathbb{N}^n \cap \mathcal{S}_{p,n}$, and let $\mathbf{v} \in \mathbb{N}^n$ with $\|\mathbf{v}\| < p$ and $A\mathbf{v} < \mathbf{r}(e, \mathbf{u})$. Set $\mathbf{u}' = \mathbf{u} + \frac{1}{p^{e+1}}\mathbf{v}$.*

(a) *If the coordinate $\eta(e, \mathbf{u})_i \geq \Omega$ for some i, then $\eta(e+1, \mathbf{u}')_i \geq \Omega$.*

(b) *Suppose that $A\mathbf{u} < \mathbf{b} + \mathbf{1}$ and there is vector $\mathbf{w} \in \frac{1}{p^h}\mathbb{N}^n \cap \mathcal{S}_{p,n}$ such that $A\mathbf{w} < \mathbf{b} + \mathbf{1}$ with $\mathbf{r}(e, \mathbf{u}) = \mathbf{r}(t, \sigma_t(\mathbf{w}))$ for some $t < h$. If $\mathbf{v} = p^{t+1}(\sigma_{t+1}(\mathbf{w}) - \sigma_t(\mathbf{w}))$, then $\mathbf{u}' \in \frac{1}{p^{e+1}}\mathbb{N}^n \cap \mathcal{S}_{p,n}$ with $A\mathbf{u}' < \mathbf{b} + \mathbf{1}$ and $\mathbf{r}(e+1, \mathbf{u}') = \mathbf{r}(t+1, \sigma_{t+1}(\mathbf{w}))$.*

(c) *Suppose $A\mathbf{u} < \mathbf{b} + \mathbf{1}$ and $\mathbf{r}(e, \mathbf{u}) = \mathbf{r}(e-s, \sigma_{e-s}(\mathbf{u}))$ for some $s \geq 1$. If $\mathbf{v} = p^{e-s+1}(\sigma_{e-s+1}(\mathbf{u}) - \sigma_{e-s}(\mathbf{u}))$, then $\mathbf{u}' \in \frac{1}{p^{e+1}}\mathbb{N}^n \cap \mathcal{S}_{p,n}$ with $A\mathbf{u}' < \mathbf{b} + \mathbf{1}$ and $\mathbf{r}(e+1, \mathbf{u}') = \mathbf{r}(e+1-s, \sigma_{e+1-s}(\mathbf{u}'))$.*

Proof. We note that $\eta(e+1, \mathbf{u}') = p(\eta(e, \mathbf{u}) - A\mathbf{v})$. Part (a) is then immediate since $\eta(e+1, \mathbf{u}')_i \geq p(\Omega - (A\mathbf{v})_i) \geq p(\Omega - (1 - \frac{1}{p})\Omega) = \Omega$ by (3.1).

(b) Since $\mathbf{w} \in \frac{1}{p^h}\mathbb{N}^n \cap \mathcal{S}_{p,n}$, we know that $\mathbf{v} \in \mathbb{N}^n$ with $\|\mathbf{v}\| < p$, and so, we have $\mathbf{u}' \in \frac{1}{p^{e+1}}\mathbb{N}^n \cap \mathcal{S}_{p,n}$. Also, because $A\sigma_{t+1}(\mathbf{w}) \leq A\mathbf{w} < \mathbf{b} + \mathbf{1}$, it easily follows that $A\mathbf{v} < \mathbf{r}(e, \sigma_t(\mathbf{w})) = \mathbf{r}(e, \mathbf{u}) \leq p^{e+1}(\mathbf{b} + \mathbf{1} - A\mathbf{u})$ so that $A\mathbf{u}' < \mathbf{b} + \mathbf{1}$.

To show that the remainder vectors $\mathbf{r}(e+1, \mathbf{u}')$ and $\mathbf{r}(t+1, \sigma_{t+1}(\mathbf{w}))$ agree, we argue coordinatewise. If $\mathbf{r}(e, \mathbf{u})_i = \Omega$, then both $\eta(e, \mathbf{u})_i, \eta(t, \sigma_t(\mathbf{w}))_i \geq \Omega$ so that $\mathbf{r}(e+1, \mathbf{u}')_i = \Omega = \mathbf{r}(t+1, \sigma_{t+1}(\mathbf{w}))_i$ by part (a). Otherwise, we must have that $\eta(e, \mathbf{u})_i = \eta(t, \sigma_t(\mathbf{w}))_i$ so that $\eta(e+1, \mathbf{u}')_i = \eta(t+1, \sigma_{t+1}(\mathbf{w}))_i$ by our observation at the beginning of the proof. Hence, it follows that $\mathbf{r}(e+1, \mathbf{u}') = \mathbf{r}(t+1, \sigma_{t+1}(\mathbf{w}))$ as claimed.

(c) This is a special case of part (b) with $\mathbf{w} = \mathbf{u}$ and $t = e - s$. $\square$

Theorem 3.12. *Suppose that λ is the output of Algorithm 3.7 for the monomial ideal I and the monomial $\mathbf{x}^{\mathbf{b}}$. Then $\lambda = \lambda_{\mathbf{b}}(I)$.*

Proof. Let $\mathbf{w} \in \mathcal{L}_\infty$ when Algorithm 3.7 terminates. Then $\mathbf{w} = \mathbf{u} + \frac{1}{p^e(p^s-1)}\mathbf{v} = \mathbf{u} + \sum_{j=1}^{\infty} \frac{1}{p^{sj+e}}\mathbf{v}$ for some $\mathbf{u} \in \mathcal{L}_e$, some $s \geq 1$ such that $\mathbf{r}(e, \mathbf{u}) = \mathbf{r}(e-s, \sigma_{e-s}(\mathbf{u}))$, and $\mathbf{v} = p^e(\mathbf{u} - \sigma_{e-s}(\mathbf{u})) \in \mathbb{N}^n$. As in Remark 3.3 and the proof of Theorem 3.4, we see that $\mathbf{u} \in \frac{1}{p^e}\mathbb{N}^n \cap \mathcal{S}_{p,n}$ and $A\mathbf{u} < \mathbf{b} + \mathbf{1}$. If we set $\mathbf{w}_h = \mathbf{u} + \sum_{j=1}^{h} \frac{1}{p^{sj+e}}\mathbf{v}$, a straightforward induction digit-by-digit starting with $\mathbf{w}_0 = \mathbf{u}$ and using part (c) of the preceding lemma shows that $\mathbf{w}_h \in \frac{1}{p^{sh+e}}\mathbb{N}^n \cap \mathcal{S}_{p,n}$ and $A\mathbf{w}_h < \mathbf{b} + \mathbf{1}$ for all h. Hence, we have $\|\mathbf{w}_h\| \leq \lambda_{\mathbf{b}}$ for all h by Theorem 2.19, and it follows that $\|\mathbf{w}\| = \lim_{h\to\infty} \|\mathbf{w}_h\| \leq \lambda_{\mathbf{b}}$. As this holds for every $\mathbf{w} \in \mathcal{L}_\infty$, this shows that $\lambda \leq \lambda_{\mathbf{b}}$.

Conversely, to see that $\lambda_{\mathbf{b}} \leq \lambda$, consider a family of witnesses $\{\mathbf{w}_e\}$ for $\lambda_{\mathbf{b}}$. Since there are only finitely many remainders $\mathbf{r}(e, \mathbf{w}_e)$, we can choose e as small as possible such that there is an $s \geq 1$ with $\mathbf{r}(e, \mathbf{w}_e) = \mathbf{r}(e-s, \mathbf{w}_{e-s})$. Now, consider the vector $\mathbf{w}' = \mathbf{w}_e + \sum_{j=1}^{\infty} \frac{1}{p^{sj+e}}\mathbf{v}'$ and the sequence of partial sums $\mathbf{w}'_h = \mathbf{w}_e + \sum_{j=1}^{h} \frac{1}{p^{sj+e}}\mathbf{v}'$, where $\mathbf{v}' = p^e(\mathbf{w}_e - \mathbf{w}_{e-s}) \in \mathbb{N}^n$. Since the vectors $\mathbf{w}_h$ for $h < e$ all have distinct remainders, Theorem 3.1 guarantees that $\mathbf{w}_h$ is added to $\mathcal{L}_h$ by Step 4 of the algorithm for all $h \leq e$ so that $\mathbf{w}'$ is appended to the set of limit vectors $\mathcal{L}_\infty$.

We will show that

$$\langle \lambda_{\mathbf{b}} \rangle_{e+sh} = \|\mathbf{w}_{e+sh}\| \leq \|\mathbf{w}'_h\| \tag{3.3}$$

for all h so that

$$\lambda_{\mathbf{b}} = \lim_{h\to\infty} \langle \lambda_{\mathbf{b}} \rangle_{e+sh} \leq \lim_{h\to\infty} \|\mathbf{w}'_h\| = \|\mathbf{w}'\| \leq \lambda.$$

This is clear for $h = 0$ since $\mathbf{w}'_0 = \mathbf{w}_e$. Suppose (3.3) holds for some h, and set $\mathbf{v} = p^{e+s(h+1)}(\mathbf{w}_{e+s(h+1)} - \mathbf{w}_{e+sh})$ and $\mathbf{v}'' = p^{e+sh}(\mathbf{w}_{e+sh} - \mathbf{w}_e)$. It suffices to show that $\|\mathbf{v}\| \leq \|\mathbf{v}'\|$ to show that $\left\|\mathbf{w}_{e+s(h+1)}\right\| \leq \left\|\mathbf{w}'_{h+1}\right\|$. Since $\mathbf{r}(e, \sigma_e(\mathbf{w}_{e+s(h+1)})) = \mathbf{r}(e, \mathbf{w}_e) = \mathbf{r}(e-s, \mathbf{w}_{e-s})$, a simple digit-by-digit induction using part (b) of the preceding lemma shows that the vector $\mathbf{w}'' = \mathbf{w}_{e-s} + \frac{1}{p^{e+s(h-1)}}\mathbf{v}'' + \frac{1}{p^{e+sh}}\mathbf{v}$ is contained in $\frac{1}{p^{e+sh}}\mathbb{N}^n \cap \mathcal{S}_{p,n}$ and satisfies $A\mathbf{w}'' < \mathbf{b} + \mathbf{1}$ so that $\|\mathbf{w}''\| \leq \langle \lambda_{\mathbf{b}} \rangle_{e+sh} = \|\mathbf{w}_{e+sh}\|$ by Theorem 2.19 and Lemma 2.22. As in the first part of the proof, we also know that $\|\mathbf{w}'_h\| \leq \lambda_{\mathbf{b}}$. Combining this with Lemma 2.22 and our inductive assumption yields

$$\begin{aligned} \|\mathbf{w}_{e-s}\| + \sum_{j=0}^{h} \frac{1}{p^{sj+e}} \|\mathbf{v}'\| = \|\mathbf{w}'{}_h\| = \langle \lambda_{\mathbf{b}} \rangle_{e+sh} \\ = \|\mathbf{w}_{e+sh}\| = \|\mathbf{w}_{e-s}\| + \frac{1}{p^e} \|\mathbf{v}'\| + \frac{1}{p^{e+sh}} \|\mathbf{v}''\| \end{aligned}$$

from which it is easily seen that $\frac{1}{p^{e+s(h-1)}}\|\mathbf{v}''\| = \sum_{j=0}^{h-1}\frac{1}{p^{sj+e}}\|\mathbf{v}'\|$ so that

$$\|\mathbf{w}_{e-s}\| + \sum_{j=0}^{h-1}\frac{1}{p^{sj+e}}\|\mathbf{v}'\| + \frac{1}{p^{e+sh}}\|\mathbf{v}\| = \|\mathbf{w}''\|$$
$$\leq \|\mathbf{w}_{e+sh}\| = \|\mathbf{w}_{e-s}\| + \sum_{j=0}^{h}\frac{1}{p^{sj+e}}\|\mathbf{v}'\|$$

implying $\|\mathbf{v}\| \leq \|\mathbf{v}'\|$ as wanted and completing the proof. □

Corollary 3.13. *Let I be a monomial ideal. Then all critical exponents of I are rational.*

Proof. This is immediate from the fact that the norm of a vector realizing the output of Algorithm 3.7 has a repeating base p expansion by construction. □

4. Examples of Computing Critical Exponents

Proposition 4.1. *Let $I = (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_n}) \subseteq S = \mathbb{F}_p[x_1, x_2, \ldots, x_m]$ be a monomial ideal of height 1. Without loss of generality, say $x_1 \mid m_i$ for all i. Suppose that $d = a_{1,1} = \min_j\{a_{1,j}\} = \max_i\{a_{i,1}\}$. Then $\operatorname{lce}(I) = 1/d$.*

Proof. If $\mathbf{w}$ is an e-witness for $\lambda = \operatorname{lce}(I)$, then the first coordinate of the inequaliy $A\mathbf{w} < \mathbf{1}$ and our assumptions on the exponent matrix imply $d\langle\lambda\rangle_e = d\|\mathbf{w}\| \leq \sum_j a_{1,j}w_j < 1$. Hence, $\langle\lambda\rangle_e < \frac{1}{d}$, and taking a limit yields $\lambda \leq \frac{1}{d}$. On the other hand, since $d\left\langle\frac{1}{d}\right\rangle_e < 1$ for all e, the vector $\mathbf{w}_e = (\left\langle\frac{1}{d}\right\rangle_e, 0, \ldots, 0) \in \mathcal{S}_{p,n}$ satisfies $A\mathbf{w}_e < \mathbf{1}$ by our assumptions on the exponent matrix so that $\left\langle\frac{1}{d}\right\rangle_e = \|\mathbf{w}_e\| \leq \lambda$ for all e by Theorem 2.19. And so, taking a limit yields $\frac{1}{d} \leq \lambda$. □

To demonstrate the tools from the previous section, we compute the Frobenius powers for a specific example to show how the characteristic can affect the results.

Example 4.2. Let $S = \mathbb{F}_p[x, y, z]$ and $I = (x^2y^2, x^3z^3)$. Then the distinct Frobenius powers of I containing I are as follows.

- If $p = 2$, then:

$$I^{[\lambda]} = \begin{cases} S, & \lambda \in [0, \frac{1}{2}) \\ (xy, xz), & \lambda \in [\frac{1}{2}, \frac{3}{4}) \\ (xy, x^2z), & \lambda \in [\frac{3}{4}, 1) \end{cases}$$

- If $p=3$, then:

$$I^{[\lambda]}=\begin{cases} S, & \lambda\in[0,\frac{1}{2}) \\ (x), & \lambda\in[\frac{1}{2},\frac{2}{3}) \\ (xy,xz), & \lambda\in[\frac{2}{3},\frac{5}{6}) \\ (xy,x^2z), & \lambda\in[\frac{5}{6},1) \end{cases}$$

- If $p\equiv 1 \pmod 6$, then:

$$I^{[\lambda]}=\begin{cases} S, & \lambda\in[0,\frac{1}{2}) \\ (x), & \lambda\in[\frac{1}{2},\frac{5}{6}) \\ (xy,x^2z), & \lambda\in[\frac{5}{6},1) \end{cases}$$

- If $p\equiv 5 \pmod 6$, then:

$$I^{[\lambda]}=\begin{cases} S, & \lambda\in[0,\frac{1}{2}) \\ (x), & \lambda\in[\frac{1}{2},\frac{5p-1}{6p}) \\ (xy,x^2z), & \lambda\in[\frac{5p-1}{6p},1) \end{cases}$$

We compute $\lambda_{(1,0,0)}$; the other computations are similar. A (family of) witnesses will be a collection of pairs $\mathbf{v}_e=(a_e,b_e)$ satisfying

$$\begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} a_e \\ b_e \end{pmatrix}<\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

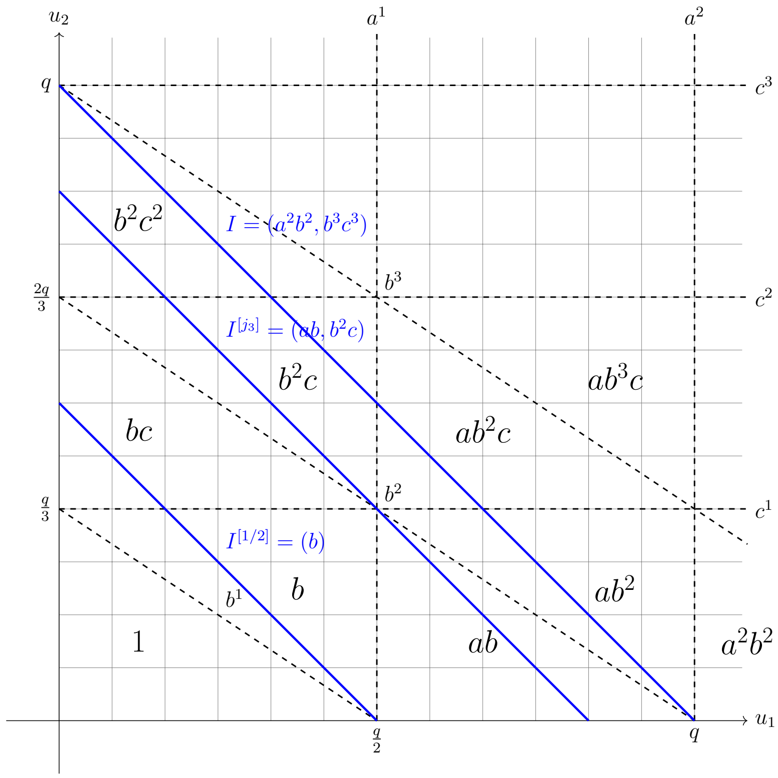
That is, $a<\frac{1}{2}$ and $b<\frac{1}{3}$ such that $a+b$ adds without carries in base p.

If $p=2$, the binary representations of $\frac{1}{2}$ and $\frac{1}{3}$ are $.0\overline{1}$ and $.\overline{01}$, respectively. Thus a_e and b_e must each have a zero in the first decimal place; the best we can do for the sum without a carry is $.0\overline{1}$. This is realized by, for example, taking $a_e=\operatorname{trunc}_e(\frac{1}{2}), b_e=0$. Thus $\lambda_{(0,1,0)}=\frac{1}{2}$, i.e., $y\in I^{[\frac{k}{q}]}$ if and only if $\frac{k}{q}<\frac{1}{2}$.

If $p=3$, the trinary representations of $\frac{1}{2}$ and $\frac{1}{3}$ are $.\overline{1}$ and $.0\overline{2}$, respectively. Thus the first digit of a_e must be at most 1 and the first digit of b_e must be a zero. Without carries, the first digit of a_e+b_e cannot be more than 1, so the best we can do for the sum is $.1\overline{2}=\frac{2}{3}$. This is realized by, for example, taking $a_e=.1=$, $b_e=\operatorname{trunc}_e(\frac{1}{3})$. Thus $\lambda_{(0,1,0)}=\frac{2}{3}$, i.e., $y\in I^{[\frac{k}{q}]}$ if and only if $\frac{k}{q}<\frac{2}{3}$.

If $p\equiv 1 \pmod 6$, set $m=p-1$, $s=\frac{m}{2}$, and $m=\frac{q}{3}$. The base p representations of $\frac{1}{2}$ and $\frac{1}{3}$ are $.\overline{s}$ and $.\overline{t}$. Since $s+t<m$, we may add these without carries. The (unique) family of witnesses is $a_e=\operatorname{trunc}_e(\frac{1}{2}), b_e=\operatorname{trunc}_e(\frac{1}{3})$. We conclude that $\lambda_{(0,1,0)}=\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$, i.e., $y\in I^{[\frac{k}{q}]}$ if and only if $\frac{k}{q}<\frac{5}{6}$.

If $p\equiv 5 \pmod 6$, set $m=p-1$, $r=p-2$, $s=\frac{m}{2}$, $t=\frac{r}{3}$, and $w=2t+1$. The base p representations of $\frac{1}{2}$ and $\frac{1}{3}$ are $.\overline{s}$ and $.\overline{tw}$. In order to add without carries, the first digit of $a_e + b_e$ must be at most $(s+t)$. But $s+w > p$, so the second digit and all subsequent digits can be arbitrary. The best we can hope for when adding without carries is $.g\overline{m}$, where $g = s+t$. One witness is $a_e = \text{trunc}_e(.\overline{s}), b_e = \text{trunc}_e(.t\overline{s})$. Thus $\lambda_{(0,1,0)} = .g\overline{m} = \frac{g+1}{p} = \frac{5p-1}{6p}$, i.e., $y \in I^{[\frac{k}{q}]}$ if and only if $\frac{k}{q} < \frac{5p-1}{6p}$.



5. Further Questions

By Corollary 2.9, every monomial ideal I containing a squarefree monomial satisfies $I^{[\lambda]} = (1)$ for all $\lambda \in (0, 1]$. This leads to some natural questions.

Question 5.1. Under what conditions do two monomial ideals I and J satisfy $I^{[\lambda]} = J^{[\lambda]}$ for all $\lambda < 1$?

In particular:

Question 5.2. Can we characterize which monomial ideals I have $\text{lce}(I) = 1$?

If every monomial of I is divisible by a p-th power, then every column of the exponent matrix A contains an entry at least p so that the only vector $\mathbf{v} \in \mathbb{N}^n$ maximizing $\|\mathbf{v}\|$ with $A\mathbf{v} < p\mathbf{1}$ in the first iteration of Algorithm 3.2 is the zero vector, meaning that $\text{lce}(I) \leq \frac{1}{p}$. This proves the converse to Corollary 2.9 when $p = 2$ and gives a restriction on the possible answers to the above question in general. However, the converse to Corollary 2.9 is false for $p \geq 3$.

Proposition 5.3. *If* $I = (\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_{p-1}}) \subseteq S = \mathbb{F}_p[x_1, x_2, \dots, x_m]$ *and* $A\mathbf{1} \leq (p-1)\mathbf{1}$*, then* $\operatorname{lce}(I) = 1$.

Proof. Using Algorithm 3.2 with $\mathbf{b} = \mathbf{0}$, we show that $\mathbf{w}_e = \frac{1}{p-1}(1 - \frac{1}{p^e})\mathbf{1}$ is an e-witness for all e. Indeed, this is clear if $e = 0$. If $e \geq 1$ and $\mathbf{w}_{e-1}$ is an $(e-1)$-witness, then the remainder vector associated to $\mathbf{w}_{e-1}$ is $\mathbf{r} = \mathbf{1} - A\mathbf{w}_{e-1}$. Since $A\mathbf{1} \leq (p-1)\mathbf{1}$, it follows that $p^e\mathbf{r} \geq p\mathbf{1}$. Hence, the vector $\mathbf{1}$ maximizes $\|\mathbf{v}\|$ among all $\mathbf{v} \in \mathbb{N}^{p-1}$ subject to the constraints $A\mathbf{v} < p^e\mathbf{r}$ and $\|\mathbf{v}\| < p$, and $\mathbf{w}_e = \mathbf{w}_{e-1} + \frac{1}{p^e}\mathbf{1}$ is an e-witness. And so, Proposition 2.24 implies that $\operatorname{lce}(I) = \lim_{e\to\infty} \|\mathbf{w}_e\| = \lim_{e\to\infty} 1 - \frac{1}{p^e} = 1$. $\square$

Using the above proposition, we can produce lots of monomial ideals I not containing a squarefree monomial with $\operatorname{lce}(I) = 1$.

Example 5.4. Suppose $p \geq 3$, and write $p - 1 = 2k$ for some integer k. Applying the preceding proposition to the exponent matrix

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ 2 & 2 & \cdots & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 2 & \cdots & 2 & 2 \\ \underbrace{0 \quad 0 \quad \cdots \quad 0}_{k} & & & & \underbrace{1 \quad \cdots \quad 1 \quad 1}_{k} \end{pmatrix}$$

shows that the $I = (xy^2, z^2w) \subseteq S = \mathbb{F}_p[x, y, z, w]$ is a monomial ideal not containing any squarefree monomials with $\operatorname{lce}(I) = 1$.

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